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On the pressure of boson and fermion systems

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Abstract. We prove that the pressure of a system of fermion particles, interacting through a generalised mean field potential, is always greater than the pressure of an identical system of boson particles at the same density and temperature, even when phase transitions occur.

1. Introduction

It is generally assumed that, except for some systems where the interaction between the particles is too strong, the pressure of a gas consisting of boson particles will always be smaller than the pressure of an identical gas of fermion particles, at the same density and temperature. This phenomenon should be due to the different statistics. Indeed, the statistics cause the bosons (fermions) to suffer a statistical attraction (repulsion). For free systems, this was indeed proved in the near classical region (high temperature, low density) (Uhlenbeck and Gropper 1932, Landau and Lifshitz 1958). Liboff (1981) was able to generalise these inequalities to all thermodynamic states of finite temperature and thus for free systems.

Here we extend these results to systems with interactions of the mean field type. Our method of proof shows clearly where the statistics come into the argument. Essentially, the method is based on comparing the density fluctuations for the boson and fermion systems. As our method does not depend in an essential way on explicit calculations, there might be some hope that an analogous approach could be used for systems with stable or superstable interactions.

In § 2, we deal with free systems. The pressure inequalities are derived both for the finite volume case and for the thermodynamic limit. The thermodynamic limit case has already been treated by Liboff, but the proof presented in this paper is essentially different.

In § 3, we prove the theorem for systems with a generalised mean field interaction. We treat systems without phase transitions of the first order, as well as systems in which phase transitions of the first order can occur. As a byproduct, we show that in the latter case the statistics also have an influence on the occurrence of the phase transition, namely that the critical temperature above which no phase transition of first order occurs is higher for the boson gas than for the fermion gas.

The ingredients for our systems are the following. Let Λ be an open bounded region of $\mathbb{R}^\nu$ with volume $V(\Lambda)$; ν is the dimension of the system. We denote by F_Λ^+ (F_Λ^-) the usual boson (fermion) Fock space, constructed on $L^2(\Lambda, \mathrm{d}x)$. The free

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Hamiltonian of the boson and the fermion system is given by

$$H^{\sigma}_{\Lambda,\mu} = \sum \varepsilon^{\sigma}_{\Lambda}(n) a^*(f^{\Lambda}_n) a(f^{\Lambda}_n) - \mu N_{\Lambda} \tag{1}$$

where $\{f^{\Lambda}_n\}_{n\geq 0}$ are the eigenfunctions and $\varepsilon^{\sigma}_{\Lambda}(n)$ the corresponding eigenvalues of the Laplacian $-\frac{1}{2}\Delta^2$ on $L^2(\Lambda, \mathrm{d}x)$ with the boundary conditions $\partial\psi/\partial n = \sigma\psi$; $a^*(f_n)$, $a(f_n)$ are the usual Fock creation and annihilation operators satisfying the canonical commutation relations in the boson case, and the canonical anticommutation relations in the fermion case. N_{Λ} is the number operator and is equal to $\Sigma\, a^*(f^{\Lambda}_n) a(f^{\Lambda}_n)$. Finally, μ is the chemical potential and $\mu < \varepsilon^{\sigma}_{\Lambda}(0)$ for the boson case, while $\mu \in \mathbb{R}$ for the fermion case. For the finite volume treatment, more general spectra $\varepsilon_{\Lambda}(n)$ can be considered, but for the thermodynamic limit we restrict for simplicity to the Laplacian. As the thermodynamic limit for free systems is independent of σ (Landau and Wilde 1979), we drop the index σ for notational convenience.

Finally we recall some definitions (see Ruelle 1969). The finite volume pressure in the grand canonical ensemble is given by

$$p_{\mathrm{B(F)}}(\Lambda, \beta, \mu) = (1/\beta V(\Lambda)) \log Z_{\mathrm{B(F)}}(\Lambda, \beta, \mu) \tag{2}$$

where the $Z_{\mathrm{B(F)}}(\Lambda, \beta, \mu)$ is the grand canonical partition function, defined by

$$Z_{\mathrm{B(F)}}(\Lambda, \beta, \mu) = \mathrm{Tr}_{F^+_{\Lambda}(F^-_{\Lambda})} \exp(-\beta H_{\Lambda,\mu}). \tag{3}$$

The density is given by

$$\rho_{\mathrm{B(F)}}(\Lambda, \beta, \mu) = \frac{1}{V(\Lambda)} \frac{\mathrm{Tr}_{F^+_{\Lambda}(F^-_{\Lambda})} N_{\Lambda} \exp(-\beta H_{\Lambda,\mu})}{\mathrm{Tr}_{F^+_{\Lambda}(F^-_{\Lambda})} \exp(-\beta H_{\Lambda,\mu})} \tag{4}$$

or

$$\rho_{\mathrm{B(F)}}(\Lambda, \beta, \mu) = \omega^{\Lambda}_{\mathrm{B(F)},\mu}(N_{\Lambda}/V(\Lambda))$$

where $\omega^{\Lambda}_{\mathrm{B(F)},\mu}$ is the grand canonical Gibbs state at the chemical potential μ and temperature $1/\beta$.

2. Pressure inequalities for free systems

2.1. The finite system

From (2), (3) and (4) it follows trivially that

$$\frac{\mathrm{d}p_{\mathrm{B(F)}}}{\mathrm{d}\mu}(\Lambda, \beta, \mu) = \rho_{\mathrm{B(F)}}(\Lambda, \beta, \mu), \tag{5}$$

$$\frac{1}{\beta} \frac{\mathrm{d}\rho_{\mathrm{B(F)}}}{\mathrm{d}\mu}(\Lambda, \beta, \mu) = \frac{\omega^{\Lambda}_{\mathrm{B(F)},\mu}(N^2_{\Lambda}) - (\omega^{\Lambda}_{\mathrm{B(F)},\mu}(N_{\Lambda}))^2}{V(\Lambda)}. \tag{6}$$

Furthermore, using (6) and the correlation inequalities (see Fannes and Verbeure 1977), we find

$$\frac{1}{\beta} \frac{\mathrm{d}\rho_{\mathrm{B}}}{\mathrm{d}\mu_{\mathrm{B}}}(\Lambda, \beta, \mu_{\mathrm{B}}) = \frac{\omega^{\Lambda}_{\mathrm{B},\mu_{\mathrm{B}}}(N_{\Lambda})}{V(\Lambda)} + \frac{1}{V(\Lambda)} \sum_{n\geq 0} (\omega^{\Lambda}_{\mathrm{B},\mu_{\mathrm{B}}}(N(f^{\Lambda}_n)))^2, \tag{7}$$

$$\frac{1}{\beta} \frac{\mathrm{d}\rho_{\mathrm{F}}}{\mathrm{d}\mu_{\mathrm{F}}}(\Lambda, \beta, \mu_{\mathrm{F}}) = \frac{\omega^{\Lambda}_{\mathrm{F},\mu_{\mathrm{F}}}(N_{\Lambda})}{V(\Lambda)} - \frac{1}{V(\Lambda)} \sum_{n\geq 0} (\omega^{\Lambda}_{\mathrm{F},\mu_{\mathrm{F}}}(N(f^{\Lambda}_n)))^2, \tag{8}$$

where $N(f^{\Lambda}_n) = a^*(f^{\Lambda}_n) a(f^{\Lambda}_n)$, $\mu_{\mathrm{F}} \in \mathbb{R}$ and $\mu_{\mathrm{B}} < \varepsilon_L(0)$.

Let

$$G_{\Lambda,\beta}(\mu_B, \mu_F) = \rho_B(\Lambda, \beta, \mu_B) - \rho_F(\Lambda, \beta, \mu_F) \qquad \forall \mu_F \in \mathbb{R}, \forall \mu_B < \varepsilon_L(0).$$

We restrict our attention to the pairs (μ_B, μ_F) for which $G_{\Lambda,\beta}(\mu_B, \mu_F) = 0$. These pairs are the chemical potentials of systems at the same density. From (7) it follows that

$$\partial G_{\Lambda,\beta}(\mu_B, \mu_F)/\partial \mu_B = \mathrm{d}\rho_B(\Lambda, \beta, \mu_B)/\mathrm{d}\mu_B \neq 0, \qquad \forall \mu_B < \varepsilon_L(0).$$

Therefore, by the implicit function theorem, there exists an analytic function f such that $\mu_B = f(\mu_F)$. We have

Theorem 2.1. For the Hamiltonian (1), and with the notations above

$$p_B(\Lambda, \beta, f(\mu_F)) < p_F(\Lambda, \beta, \mu_F) \qquad \forall \mu_F \in \mathbb{R}, \forall \beta > 0.$$

Proof. First remark that

$$\lim_{\mu_F \to -\infty} f(\mu_F) = -\infty, \qquad \lim_{\mu_F \to -\infty} p_F(\Lambda, \beta, \mu_F) = \lim_{\mu_F \to -\infty} p_B(\Lambda, \beta, f(\mu_F)) = 0.$$

Therefore, it is sufficient to prove that

$$A(\mu_F) \equiv \frac{\mathrm{d}}{\mathrm{d}\mu_F}(p_F(\Lambda, \beta, \mu_F) - p_B(\Lambda, \beta, f(\mu_F))) > 0, \qquad \forall \mu_F \in \mathbb{R}.$$

But

$$A(\mu_F) = \rho_F(\Lambda, \beta, \mu_F)(1 - \mathrm{d}f(\mu_F)/\mathrm{d}\mu_F).$$

From (7) and (8)

$$\frac{\mathrm{d}f(\mu_F)}{\mathrm{d}\mu_F} = \frac{\mathrm{d}\rho_F(\Lambda, \beta, \mu)/\mathrm{d}\mu|_{\mu_F}}{\mathrm{d}\rho_B(\Lambda, \beta, \mu)/\mathrm{d}\mu|_{f(\mu_F)}} < 1.$$

Therefore $A(\mu_F) > 0$, $\forall \mu_F \in \mathbb{R}$.

Analogously, one can prove the stronger results:

(i) $p_B(\Lambda, \beta, \mu_B) < \beta^{-1}\rho_B(\Lambda, \beta, \mu_B) \qquad \forall \mu_B < \varepsilon_L(0)$;

(ii) $\beta^{-1}\rho_F(\Lambda, \beta, \mu_F) < p_F(\Lambda, \beta, \mu_F) \qquad \forall \mu_F \in \mathbb{R}$.

2.2. *The thermodynamic limit*

Denote by Λ_L the centred cube with side L. Define for all $\beta > 0$, for all $\mu < 0$ in the boson case and for all $\mu \in \mathbb{R}$ in the fermion case,

$$p_{B(F)}(\beta, \mu) = \lim_{L \to \infty} p_{B(F)}(\Lambda_L, \beta, \mu), \qquad \rho_{B(F)}(\beta, \mu) = \lim_{L \to \infty} \rho_{B(F)}(\Lambda_L, \beta, \mu).$$

These limits exist and are independent of the boundary conditions (Landau and Wilde 1979, Ruelle 1969). It is well known that for all $\mu < 0$, $\rho_B(\beta, \mu) < \rho_c$, where ρ_c is the critical density given by

$$\rho_c = \int \frac{\mathrm{d}^\nu k}{(2\pi)^\nu} \frac{1}{\exp(\frac{1}{2}\beta k^2) - 1} \qquad \text{for } \nu \geq 3,$$
$$= +\infty \qquad \text{for } \nu = 1, 2. \tag{9}$$

Using Griffiths' lemma (see e.g. Hepp and Lieb 1973), one finds

$$\mathrm{d}p_{\mathrm{B(F)}}/\mathrm{d}\mu\ (\beta,\mu)=\rho_{\mathrm{B(F)}}(\beta,\mu).$$

Furthermore, using the explicit expressions for $\rho_{\mathrm{B(F)}}(\beta,\mu)$ given by

$$\rho_{\mathrm{B}}(\beta,\mu)=\int\frac{\mathrm{d}^\nu k}{(2\pi)^\nu}\frac{1}{\exp[\beta(k^2/2-\mu)]-1}\qquad\forall\mu<0,\forall\beta>0,\qquad(10)$$

$$\rho_{\mathrm{F}}(\beta,\mu)=\int\frac{\mathrm{d}^\nu k}{(2\pi)^\nu}\frac{1}{\exp[\beta(k^2/2-\mu)]+1}\qquad\forall\mu\in\mathbb{R},\forall\beta>0\qquad(11)$$

(see e.g. Bratelli and Robinson 1981), one can prove, completely analogous to the finite volume case, that

(i) μ_{B} can be written as a function of μ_{F}, or $\mu_{\mathrm{B}}=f(\mu_{\mathrm{F}})$, $\forall\mu_{\mathrm{F}}$ such that $\rho_{\mathrm{F}}(\beta,\mu_{\mathrm{F}})<\rho_{\mathrm{c}}$ or $\forall\mu_{\mathrm{F}}:f(\mu_{\mathrm{F}})<0$;

(ii) $\mathrm{d}f(\mu_{\mathrm{F}})/\mathrm{d}\mu_{\mathrm{F}}<1$ implying that

$$p_{\mathrm{B}}(\beta,f(\mu_{\mathrm{F}}))<p_{\mathrm{F}}(\beta,\mu_{\mathrm{F}}),\qquad\forall\mu_{\mathrm{F}}:f(\mu_{\mathrm{F}})<0.\qquad(12)$$

Next consider the case where the density of the systems is greater than ρ_{c}. Let $\mu(\Lambda_L)$ be defined by $\rho=\rho_{\mathrm{B}}(\Lambda_L,\beta,\mu(\Lambda_L))$; then it is well known that $\lim_{L\to\infty}\mu(\Lambda_L)=0$, for $\rho\geqslant\rho_{\mathrm{c}}$ and

$$\lim_{L\to\infty}p_{\mathrm{B}}(\Lambda_L,\beta,\mu(\Lambda_L))=p_{\mathrm{B}}(\beta,0)\equiv\lim_{\mu_{\mathrm{B}}\uparrow0}p_{\mathrm{B}}(\beta,\mu_{\mathrm{B}})$$

(see e.g. Ziff *et al* 1977).

Define $f^{-1}(0)$ as

$$f^{-1}(0)=\lim_{\mu_{\mathrm{B}}\uparrow0}f^{-1}(\mu_{\mathrm{B}})\qquad\text{and}\qquad f(\mu_{\mathrm{F}})=0,\ \forall\mu_{\mathrm{F}}\geqslant f^{-1}(0).$$

Now, since $\mu_{\mathrm{F}}\to p_{\mathrm{B}}(\beta,f(\mu_{\mathrm{F}}))$ is constant on $\mu_{\mathrm{F}}\geqslant f^{-1}(0)$, while $\mu_{\mathrm{F}}\to p_{\mathrm{F}}(\beta,\mu_{\mathrm{F}})$ is increasing for all μ_{F}, it is sufficient to prove that $p_{\mathrm{B}}(\beta,0)<p_{\mathrm{F}}(\beta,f^{-1}(0))$ to obtain the inequality

$$p_{\mathrm{B}}(\beta,f(\mu_{\mathrm{F}}))<p_{\mathrm{F}}(\beta,\mu_{\mathrm{F}})\qquad\text{for all }\mu_{\mathrm{F}}\geqslant f^{-1}(0).$$

From (12)

$$p_{\mathrm{B}}(\beta,0)\leqslant p_{\mathrm{F}}(\beta,f^{-1}(0)).\qquad(13)$$

To prove that (13) is in fact a strict inequality, we look for a strictly positive lower bound for $p_{\mathrm{F}}(\beta,f^{-1}(0))-p_{\mathrm{B}}(\beta,0)$. The argument here is a very simple one and is given only for completeness. Let $f(\mu_{\mathrm{F}})<0$ and take $\mu_{\mathrm{F}}>-A$; then

$$\begin{aligned}p_{\mathrm{F}}(\beta,\mu_{\mathrm{F}})-p_{\mathrm{B}}(\beta,f(\mu_{\mathrm{F}}))&=\int_{-\infty}^{\mu_{\mathrm{F}}}\rho_{\mathrm{F}}(\beta,\mu)\left(1-\frac{\mathrm{d}f(\mu)}{\mathrm{d}\mu}\right)\mathrm{d}\mu\\&\geqslant\int_{-A}^{\mu_{\mathrm{F}}}\rho_{\mathrm{F}}(\beta,\mu)\left(1-\frac{\mathrm{d}f(\mu)}{\mathrm{d}\mu}\right)\mathrm{d}\mu\\&\geqslant\int_{-A}^{\mu_{\mathrm{F}}}\mathrm{d}\mu\int\frac{\mathrm{d}^\nu k}{(2\pi)^\nu}\frac{1}{\{\exp[\beta(k^2/2-\mu)]+1\}^2}.\end{aligned}$$

As $f(\mu_{\mathrm{F}})<\mu_{\mathrm{F}}$ for all μ_{F}, we thus have

$$\lim_{\mu_{\mathrm{F}}\uparrow f^{-1}(0)}(p_{\mathrm{F}}(\beta,\mu_{\mathrm{F}})-p_{\mathrm{B}}(\beta,f(\mu_{\mathrm{F}})))\geqslant A\int\frac{\mathrm{d}^\nu k}{(2\pi)^\nu}\frac{1}{\{\exp[\beta(k^2/2+A)]+1\}^2}.$$

Theorem 2.2. Under the assumptions above

$$p_{\mathrm{B}}(\beta, f(\mu_{\mathrm{F}})) < p_{\mathrm{F}}(\beta, \mu_{\mathrm{F}}) \qquad \forall \mu_{\mathrm{F}} \in \mathbb{R}, \forall \beta > 0.$$

Moreover, as in the finite volume case, one can also prove that

$$p_{\mathrm{B}}(\beta, f(\mu_{\mathrm{F}})) < \frac{1}{\beta} \rho_{\mathrm{B}}(\beta, f(\mu_{\mathrm{F}})) = \frac{1}{\beta} \rho_{\mathrm{F}}(\beta, \mu_{\mathrm{F}}) < p_{\mathrm{F}}(\beta, \mu_{\mathrm{F}}) \qquad \forall \mu_{\mathrm{F}} \in \mathbb{R}, \forall \beta > 0.$$

Recently theorem 2.2. was proved in Liboff (1981) by an explicit calculation of the pressures. Our proof shows moreover that the result is a direct consequence of the statistics.

3. Pressure inequalities for imperfect systems

We only consider systems in the thermodynamic limit. The Hamiltonian for the finite fermion and boson system is given by

$$H_{\Lambda_L,\mu} = \sum_{n \geq 0} \varepsilon_{\Lambda_L}(n) a^*(f_n^{\Lambda_L}) a(f_n^{\Lambda_L}) - \mu N_{\Lambda_L} + V(\Lambda_L) F\left(\frac{N_{\Lambda_L}}{V(\Lambda_L)}\right). \tag{14}$$

The $\varepsilon_{\Lambda_L}(n)$ are defined in the introduction (for simplicity, only Dirichlet boundary conditions are considered). F is a continuously differentiable function satisfying $F(0) = 0$ and $\lim_{x \to \infty} F'(x) = \infty$. This last condition guarantees that $\exp(-\beta H_{\Lambda_L,\mu})$ is trace class for all values of $\mu \in \mathbb{R}$ and all Λ_L.

We are interested in the density as a function of β and μ. First consider the boson gas. Let μ_0 be any strictly negative number and let the map γ_{B} be defined on $\mathbb{R}^+$ by

$$\gamma_{\mathrm{B}}(x) = (\mu_{\mathrm{B}}^{\mathrm{I}}(x) - \mu_0)x + \beta^{-1} \int \frac{\mathrm{d}^\nu k}{(2\pi)^\nu} \log\{1 - \exp[-\beta(\tfrac{1}{2}k^2 - \mu_{\mathrm{B}}^{\mathrm{I}}(x))]\}$$
$$- \beta^{-1} \int \frac{\mathrm{d}^\nu k}{(2\pi)^\nu} \log\{1 - \exp[-\beta(\tfrac{1}{2}k^2 - \mu_0)]\} \tag{15}$$

where $\mu_{\mathrm{B}}^{\mathrm{I}}(x)$ is defined by

$$x = \int \frac{\mathrm{d}^\nu k}{(2\pi)^\nu} \frac{1}{\exp[\beta(\frac{1}{2}k^2 - \mu_{\mathrm{B}}^{\mathrm{I}}(x))] - 1} \qquad \text{if } x < \rho_{\mathrm{c}}, \tag{16}$$

$$\mu_{\mathrm{B}}^{\mathrm{I}}(x) = 0 \qquad \text{if } x \geq \rho_{\mathrm{c}} \tag{17}$$

(ρ_{c} is the critical density for boson condensation defined by (9)). Furthermore, define $\bar{\rho}_{\mathrm{B}}(\beta, \mu)$ as the smallest values of x for which the function $\gamma_{\mathrm{B}}(x) + F(x) + (\mu_0 - \mu)x$ takes its minimal value.

Then, as Davies (1972) proved,

$$\lim_{L \to \infty} \rho_{\mathrm{B}}(\Lambda_L, \beta, \mu) = \bar{\rho}_{\mathrm{B}}(\beta, \mu)$$

for all points μ of continuity of the map $\mu \to \bar{\rho}_{\mathrm{B}}(\beta, \mu)$, and for all these points, $\bar{\rho}_{\mathrm{B}}(\beta, \mu)$ is the unique density of the system at the chemical potential μ.

The technique of Davies can also be used for imperfect fermion systems, to obtain an expression for the density in terms of β and μ. For any $\mu_0 \in \mathbb{R}$, let $\gamma_{\mathrm{F}}(x)$ be given by

$$\gamma_F(x) = (\mu_F^I(x) - \mu_0)x - \beta^{-1} \int \frac{d^\nu k}{(2\pi)^\nu} \log\{1 + \exp[-\beta(\tfrac{1}{2}k^2 - \mu_F^I(x))]\}$$
$$+\beta^{-1} \int \frac{d^\nu k}{(2\pi)^\nu} \log\{1 + \exp[-\beta(\tfrac{1}{2}k^2 - \mu_0)]\} \qquad \forall x \in \mathbb{R}^+ \tag{18}$$

where $\mu_F^I(x)$ is defined by

$$x = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{\exp[\beta(\frac{1}{2}k^2 - \mu_F^I(x))] + 1}. \tag{19}$$

Again, if $\bar{\rho}_F(\beta, \mu)$ is the smallest value of x for which the function $\gamma_F(x) + F(x) + (\mu_0 - \mu)x$ reaches its minimal value, $\lim_{L\to\infty} \rho_F(\Lambda_L, \beta, \mu) = \bar{\rho}_F(\beta, \mu)$ for all points μ where the map $\mu \to \bar{\rho}_F(\beta, \mu)$ is continuous.

Remark that $\gamma_B(x)$ and $\mu_B(x)$ are written out in a slightly different way than in Davies (1972), but it can easily be seen that they are equivalent. Also, although Davies derives his result only in three dimensions, his technique can be used to derive results in arbitrary dimensions.

In § 3.1, we treat the case where F is a twice continuously differentiable, strictly convex function. In this case no phase transition of first order (i.e. no discontinuity of $\bar{\rho}_{B(F)}(\beta, \mu)$ as a function of μ) occurs. In § 3.2, we treat the case where F is not necessarily strictly convex, implying that phase transitions of first order can occur.

3.1. Imperfect systems without phase transitions of the first order

Here, we consider systems with a Hamiltonian as in (14), where the function F is twice continuously differentiable and strictly convex. In this case, the existence of the thermodynamic limit for the pressure for both the boson and fermion gas is guaranteed by:

(i) translation invariance;

(ii) uniform boundedness of the grand canonical partition function Z_Λ;

(iii) $Z_{\Lambda_1} \cup Z_{\Lambda_2} \geq Z_{\Lambda_1} Z_{\Lambda_2}$ for each Λ_1 and Λ_2 such that $\Lambda_1 \cap \Lambda_2 = \varnothing$ (for more details see Ruelle 1969). This follows from the inequalities

$$(V_1 + V_2)F\left(\frac{n_1 + n_2}{V_1 + V_2}\right) \leq V_1 F\left(\frac{n_1}{V_1}\right) + V_2 F\left(\frac{n_2}{V_2}\right) \qquad \text{(convexity of } F)$$

and

$$Q_{\Lambda_1 \cup \Lambda_2}(n, \beta) \geq \sum_{m=0}^{n} Q_{\Lambda_1}(m, \beta) Q_{\Lambda_2}(n - m, \beta)$$

where $Q_\Lambda(n, \beta)$ is the canonical partition function for the free system.

Proposition 3.1. For the given Hamiltonian (14), $\bar{\rho}_B$ is defined implicitly as a function of β and μ, by

(i) if $\mu < F'(\rho_c)$, then

$$\bar{\rho}_B(\beta, \mu) = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{\exp[\beta(\frac{1}{2}k^2 - \mu + F'(\bar{\rho}_B(\beta, \mu)))] - 1}; \tag{20}$$

(ii) if $\mu \geqslant F'(\rho_c)$, then

$$F'(\bar\rho_B(\beta,\mu)) = \mu. \tag{21}$$

Proof. First remark that, as F is strictly convex, the function $x \to \gamma_B(x) + F(x) + (\mu_0 - \mu)x$ is also strictly convex and therefore reaches its minimum at the point where the derivative vanishes.

(i) Suppose first that $\mu < F'(\rho_c)$. In this case, one can prove that $\bar\rho_B(\beta,\mu) < \rho_c$. Indeed, for any $x_0 \geqslant \rho_c$, $F'(x_0) \geqslant F'(\rho_c)$ as F is convex. So, $F'(x_0) - \mu \geqslant F'(\rho_c) - \mu > 0$ and therefore $\gamma_B(x) + F(x) + (\mu_0 - \mu)x$ cannot take its minimum value at x_0. Hence, the minimum lies in the region $x \leqslant \rho_c$ and $\bar\rho_B(\beta,\mu)$ is the point where

$$F'(x) + \mu_B^I(x) - \mu = 0, \qquad \text{or} \qquad \mu_B^I(\bar\rho_B(\beta,\mu)) = -F'(\bar\rho_B(\beta,\mu)) + \mu. \tag{22}$$

Combining (22) and (16), we find

$$\bar\rho_B(\beta,\mu) = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{\exp[\beta(\frac{1}{2}k^2 - \mu + F'(\bar\rho_B(\beta,\mu)))] - 1} \qquad \text{for } \mu < F'(\rho_c).$$

(ii) Suppose that $\mu \geqslant F'(\rho_c)$. Then, by an analogous argument we find that $\bar\rho_B(\beta,\mu) \geqslant F'(\rho_c)$. Therefore, $\bar\rho_B(\beta,\mu)$ is the point x where $F'(x) - \mu = 0$ or $F'(\bar\rho_B(\beta,\mu)) = \mu$ for $\mu \geqslant F'(\rho_c)$.

Proposition 3.2. For the imperfect fermion system, $\bar\rho_F$ is defined implicitly as a function of β and μ by

$$\bar\rho_F(\beta,\mu) = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{\exp[\beta(\frac{1}{2}k^2 - \mu + F'(\rho_F(\beta,\mu)))] + 1} \qquad \forall \mu \in \mathbb{R}. \tag{23}$$

Proof. The proof is completely analogous to the proof of proposition 3.1.

Remark. In the case of strictly convex F, the function $\mu \to \bar\rho_{B(F)}(\beta,\mu)$ is everywhere continuous, implying that $\bar\rho_{B(F)}(\beta,\mu)$ is the density for all $\mu \in \mathbb{R}$.

Again we want to find μ_B as a function of μ_F by equating the densities. This follows from:

Lemma 3.3. With $\bar\rho_B$ and $\bar\rho_F$ as defined in propositions 3.1 and 3.2, one has: if $g_1(\mu)$ and $g_2(\mu)$ are defined by

$$g_1(\mu) = \beta \int \frac{d^\nu k}{(2\pi)^\nu} \frac{\exp[\beta(\frac{1}{2}k^2 - \mu)]}{\{\exp[\beta(\frac{1}{2}k^2 - \mu)] + 1\}^2},$$

$$g_2(\mu) = \beta \int \frac{d^\nu k}{(2\pi)^\nu} \frac{\exp[\beta(\frac{1}{2}k^2 - \mu)]}{\{\exp[\beta(\frac{1}{2}k^2 - \mu)] - 1\}^2} \qquad (\mu < 0),$$

then

(i) $$\frac{d\bar\rho_F}{d\mu}(\beta,\mu) = \frac{g_1(\mu - F'(\bar\rho_F(\beta,\mu)))}{1 + F''(\bar\rho_F(\beta,\mu))g_1(\mu - F'(\bar\rho_F(\beta,\mu)))};$$

(ii) if $\bar\rho_B(\beta,\mu) < \rho_c$:

$$\frac{d\bar\rho_B}{d\mu}(\beta,\mu) = \frac{g_2(\mu - F'(\bar\rho_B(\beta,\mu)))}{1 + F''(\bar\rho_B(\beta,\mu))g_2(\mu - F'(\bar\rho_B(\beta,\mu)))};$$

if $\bar{\rho}_B(\beta, \mu) > \rho_c$:

$$\frac{d\bar{\rho}_B}{d\mu}(\beta, \mu) = \frac{1}{F''(\bar{\rho}_B(\beta, \mu))}.$$

Proof. We only prove (ii).

(a) $\bar{\rho}_B(\beta, \mu) < \rho_c$. In this case

$$\bar{\rho}_B(\beta, \mu) = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{\exp[\beta(\frac{1}{2}k^2 - \mu + F'(\bar{\rho}_B(\beta, \mu)))] - 1}$$

and

$$\frac{d}{d\mu}\bar{\rho}_B(\beta, \mu) = \beta\left(\int \frac{d^\nu k}{(2\pi)^\nu} \frac{\exp[\beta(\frac{1}{2}k^2 - \mu + F'(\bar{\rho}_B(\beta, \mu)))]}{\{\exp[\beta(\frac{1}{2}k^2 - \mu + F'(\bar{\rho}_B(\beta, \mu)))] - 1\}^2}\right) \times\left(1 - F''(\bar{\rho}_B(\beta, \mu))\frac{d}{d\mu}\bar{\rho}_B(\beta, \mu)\right).$$

So,

$$\frac{d}{d\mu}\bar{\rho}_B(\beta, \mu)(1 + F''(\bar{\rho}_B(\beta, \mu))g_2(\mu - F'(\bar{\rho}_B(\beta, \mu)))) = g_2(\mu - F'(\bar{\rho}_B(\beta, \mu)))$$

from which the result follows.

(b) $\bar{\rho}_B(\beta, \mu) > \rho_c$. Then, $F'(\bar{\rho}_B(\beta, \mu)) = \mu$ and

$$\frac{d}{d\mu}F'(\bar{\rho}_B(\beta, \mu)) = F''(\bar{\rho}_B(\beta, \mu))\frac{d\bar{\rho}_B(\beta, \mu)}{d\mu} = 1.$$

The existence of a function f relating the chemical potentials $\mu_B = f(\mu_F)$ is now immediate from the implicit function theorem and lemma 3.3.

Theorem 3.4. For the Hamiltonian (14), and for equal densities

$$p_B(\beta, f(\mu_F)) < p_F(\beta, \mu_F) \qquad \forall \mu_F \in \mathbb{R}, \forall \beta > 0. \tag{24}$$

Proof. We distinguish the cases (i) $\mu_F \in \mathbb{R}$ such that $f(\mu_F) < F'(\rho_c)$, (ii) $\mu_F \in \mathbb{R}$ such that $f(\mu_F) > F'(\rho_c)$, (iii) $\mu_F \in \mathbb{R}$ such that $f(\mu_F) = F'(\rho_c)$.

(i) First, consider μ_F such that $f(\mu_F) < F'(\rho_c)$. Then $\bar{\rho}_B(\beta, f(\mu_F)) = \bar{\rho}_F(\beta, \mu_F) = \rho < \rho_c$. As in § 2, it is sufficient to prove that

$$df(\mu_F)/d\mu_F < 1.$$

Let μ_F^I and μ_B^I be the chemical potentials of the ideal systems such that

$$\rho_F^I(\beta, \mu_F^I) = \rho_B^I(\beta, \mu_B^I) = \rho$$

where $\rho_{B(F)}^I$ are as in (10) and (11). Comparing (11) and (23), one finds

$$\mu_F^I = \mu_F - F'(\bar{\rho}_F(\beta, \mu_F)).$$

Comparing (10) and (20), one finds

$$\mu_B^I = f(\mu_F) - F'(\bar{\rho}_B(\beta, f(\mu_F))).$$

So,

$$g_1(\mu_F - F'(\bar\rho_F(\beta, \mu_F))) = g_1(\mu_F^I) < \rho_F^I(\beta, \mu_F^I) = \rho, \tag{25}$$

$$g_2(f(\mu_F) - F'(\bar\rho_B(\beta, f(\mu_F)))) = g_2(\mu_B^I) > \rho_B^I(\beta, \mu_B^I) = \rho. \tag{26}$$

Finally, remark that the map $x \in]-\lambda^{-1}, \infty[\to x/(1+\lambda x)$ is increasing for all $\lambda \in \mathbb{R}$. Then, it follows from lemma 3.3 and from (25) and (26) that

$$\frac{df(\mu_F)}{d\mu_F} = \frac{d\bar\rho_F(\beta, \mu)/d\mu|_{\mu_F}}{d\bar\rho_B(\beta, \mu)/d\mu|_{f(\mu_F)}} < 1 \tag{27}$$

for all μ_F such that $f(\mu_F) < F'(\rho_c)$.

(ii) Consider any μ_F such that $f(\mu_F) > F'(\rho_c)$. Then

$$\bar\rho_F(\beta, \mu_F) = \bar\rho_B(\beta, f(\mu_F)) > \rho_c.$$

From lemma 3.3,

$$\begin{aligned}\frac{d\bar\rho_B}{d\mu}(\beta, \mu)\bigg|_{f(\mu_F)} &= \frac{1}{F''(\bar\rho_B(\beta, f(\mu_F)))} \\ &> \frac{g_1(\mu_F - F'(\bar\rho_F(\beta, \mu_F)))}{1 + F''(\bar\rho_F(\beta, \mu_F))g_1(\mu_F - F'(\bar\rho_F(\beta, \mu_F)))} = \frac{d\bar\rho_F}{d\mu}(\beta, \mu)\bigg|_{\mu_F}.\end{aligned} \tag{28}$$

Again,

$$df(\mu_F)/d\mu_F < 1$$

for all μ_F such that $f(\mu_F) > F'(\rho_c)$.

(iii) Special care should be taken if $f(\mu_F) = F'(\rho_c)$, since in this case

$$\bar\rho_F(\beta, \mu_F) = \bar\rho_B(\beta, f(\mu_F)) = \rho_c.$$

However, this can be treated as the ideal gas; see proof of theorem 2.2. One finds

$$\bar\rho_B(\beta, f(\mu_F)) < \bar\rho_F(\beta, \mu_F) \qquad \text{for } \mu_F \text{ such that } f(\mu_F) = F'(\rho_c). \tag{29}$$

The theorem follows from (27), (28) and (29).

3.2. Imperfect systems with phase transitions of the first order

Let us now consider imperfect systems, with a Hamiltonian given by (14), where the function $F: x \in \mathbb{R}^+ \to F(x)$ is twice continuously differentiable, but not necessarily strictly convex. Then phase transitions of the first order can occur (see Davies 1972). Again, only Dirichlet boundary conditions are considered. In the first lemma, we prove the existence of the thermodynamic limit for the pressure.

Lemma 3.5. If there exists a constant $c \in \mathbb{R}$, such that $\forall x \in \mathbb{R}^+$, $F(x) \geq cx$ then, with the notations above,

$$p_{B(F)}(\beta, \mu) \equiv \lim_{L\to\infty} p_{B(F)}(\Lambda_L, \beta, \mu) = \int_{-\infty}^{\mu} \bar\rho_{B(F)}(\beta, \mu)\, d\mu.$$

Proof. It is sufficient to prove that

$$\lim_{n\to\infty} p_{B(F)}(\Lambda_n, \beta, \mu) = \int_{-\infty}^{\mu} \bar\rho_{B(F)}(\beta, \mu)\, d\mu$$

for any sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ of cubes Λ_n with increasing side. We know, as Davies (1972) proved, that

$$\lim_{n\to\infty} \rho_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu) = \bar{\rho}_{\mathrm{B(F)}}(\beta, \mu)$$

for all $\mu \in \mathbb{R}$ for which the function $\mu \in \mathbb{R} \to \bar{\rho}_{\mathrm{B(F)}}(\beta, \mu)$ is continuous and that there are only a countable number of discontinuity points for this function. Therefore

$$\lim_{n\to\infty} \sup \rho_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu) = \lim_{n\to\infty} \inf \rho_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu) = \bar{\rho}_{\mathrm{B(F)}}(\beta, \mu) \text{ AE.} \tag{30}$$

Now, denote by $p^{\mathrm{I}}_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu)$ the pressure for the ideal boson (fermion) gas and take $\mu_0 \in \mathbb{R}$ such that $\mu_0 - c < 0$. For this μ_0, we prove that

$$\lim_{n\to\infty} p_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu_0) = \int_{-\infty}^{\mu_0} \bar{\rho}_{\mathrm{B(F)}}(\beta, \mu)\, \mathrm{d}\mu.$$

As there exists a constant c such that $F(x) \geqslant cx$ for all $x \in \mathbb{R}^+$, it follows easily that

$$p_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu_0) \leqslant p^{\mathrm{I}}_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu_0 - c).$$

Moreover, since the sequence $\{p^{\mathrm{I}}_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu)\}_{n\in\mathbb{N}}$ is convergent, it is bounded or

$$\exists A \in \mathbb{R}^+ : p^{\mathrm{I}}_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu_0 - c) \leqslant A, \qquad \forall n \in \mathbb{N}.$$

We prove consecutively that

$$\begin{aligned}\int_{-\infty}^{\mu_0} \bar{\rho}_{\mathrm{B(F)}}(\beta, \mu)\, \mathrm{d}\mu &\leqslant \lim_{n\to\infty} \inf \int_{-\infty}^{\mu_0} \rho_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu)\, \mathrm{d}\mu \\ &\leqslant \lim_{n\to\infty} \sup \int_{-\infty}^{\mu_0} \rho_{\mathrm{B(F)}}(\Lambda_n, \beta, \mu)\, \mathrm{d}\mu \\ &\leqslant \int_{-\infty}^{\mu_0} \bar{\rho}_{\mathrm{B(F)}}(\beta, \mu)\, \mathrm{d}\mu.\end{aligned} \tag{31}$$

Define $g_n(\mu) = \inf_{m\in\mathbb{N}} \rho_{\mathrm{B(F)}}(\Lambda_{n+m}, \beta, \mu)$, $\forall \mu < \mu_0$. Then $g_n(\mu) \leqslant \rho_{\mathrm{B(F)}}(\Lambda_{n+m}, \beta, \mu)$, $\forall m \in \mathbb{N}$, $\forall \mu < \mu_0$, and

$$\begin{aligned}\int_{-\infty}^{\mu_0} g_n(\mu)\, \mathrm{d}\mu &\leqslant \int_{-\infty}^{\mu_0} \rho_{\mathrm{B(F)}}(\Lambda_{n+m}, \beta, \mu)\, \mathrm{d}\mu \\ &= p_{\mathrm{B(F)}}(\Lambda_{n+m}, \beta, \mu_0) \leqslant A, \qquad \forall m \in \mathbb{N}\end{aligned} \tag{32}$$

implying

$$\sup_{n\in\mathbb{N}} \int_{-\infty}^{\mu_0} g_n(\mu)\, \mathrm{d}\mu \leqslant A.$$

Furthermore,

$$\sup_{n\in\mathbb{N}} \int_{-\infty}^{\mu_0} g_n(\mu)\, \mathrm{d}\mu = \int_{-\infty}^{\mu_0} \sup_{n\in\mathbb{N}} g_n(\mu)\, \mathrm{d}\mu \tag{33}$$

as $g_{n_2} \geqslant g_{n_1}$ if $n_2 > n_1$ (see Dieudonné 1969). From (32), it follows immediately that

$$\int_{-\infty}^{\mu_0} g_n(\mu)\, \mathrm{d}\mu \leqslant \inf_{m\in\mathbb{N}} \int_{-\infty}^{\mu_0} \rho_{\mathrm{B(F)}}(\Lambda_{n+m}, \beta, \mu)\, \mathrm{d}\mu.$$

Taking the supremum over n and using (33) and (30), we find

$$\int_{-\infty}^{\mu_0} \liminf_{n\to\infty} \rho_{B(F)}(\Lambda_n, \beta, \mu)\, d\mu = \int_{-\infty}^{\mu_0} \bar{\rho}_{B(F)}(\beta, \mu)\, d\mu$$
$$\leq \liminf_{n\to\infty} \int_{\infty}^{\mu} \rho_{B(F)}(\Lambda_n, \beta, \mu)\, d\mu \leq A.$$

This proves the first inequality of (31). The second inequality being trivial, the last inequality follows from an analogous argument as the first. This proves that

$$\lim_{n\to\infty} p_{B(F)}(\Lambda_n, \beta, \mu_0) = \int_{-\infty}^{\mu_0} \bar{\rho}_{B(F)}(\beta, \mu)\, d\mu.$$

The existence of the thermodynamic limit for the pressure for all $\mu \in \mathbb{R}$ follows from a direct application of the dominated convergence theorem on

$$p_{B(F)}(\Lambda_n, \beta, \mu) - p_{B(F)}(\Lambda_n, \beta, \mu_0) = \int_{\mu_0}^{\mu} \rho_{B(F)}(\Lambda_n, \beta, \mu)\, d\mu.$$

An interesting function F is $F(x) = -ax^2 + bx^4$ $(a, b > 0)$. For this function, one can indeed find a constant c such that $F(x) \geq cx$ for all $x \in \mathbb{R}^+$, e.g.

$$c = -(a^3/2b)^{1/2}.$$

Let us denote the functions $\gamma_{B(F)}(x) + F(x) + (\mu_0 - \mu)x$ by $A_{B(F)}(x, \mu)$. Remember that $\bar{\rho}_{B(F)}(\beta, \mu)$ is the lowest value of x for which the function $A_{B(F)}(x, \mu)$ attains its minimum. The function $\mu \to \bar{\rho}_{B(F)}(\beta, \mu)$ is easily seen to be increasing and is everywhere continuous except at those points μ for which the function attains its minimum value at more than one point. In the latter case, $\bar{\rho}_{B(F)}(\beta, \mu)$ shows a jump from the lowest value of x to the highest value of x for which the minimum is attained. We know that $\bar{\rho}_{B(F)}(\beta, \mu)$ gives the density if μ is a point of continuity. We still have to say what happens at a discontinuity point. Let μ be a discontinuity point such that $\bar{\rho}_{B(F)}(\beta, \mu)$ jumps from ρ_1 to ρ_2 at μ. Define μ_L by $\rho_{B(F)}(\Lambda_L, \beta, \mu_L) = \rho$ for all $L \in \mathbb{R}^+$, where ρ is a given density in the interval $[\rho_1, \rho_2]$. It follows that $\lim_{L\to\infty} \mu_L = \mu$. Furthermore, in the thermodynamic limit, the system splits up in two phases, one with density ρ_1 and one with density ρ_2, and this in such a proportion that the mean density is ρ (Davies 1972).

In the next theorem, we frequently use the fact that $x \in \mathbb{R}^+ \to \mu_F^1(x) - \mu_B^1(x)$ is increasing, which can easily be proved.

Theorem 3.6. Consider boson and fermion systems with local Hamiltonians given by

$$H_{\Lambda_L,\mu} = \sum_{n\geq 0} \varepsilon_{\Lambda_L}(n) a^*(f_n^{\Lambda_L}) a(f_n^{\Lambda_L}) - \mu N_{\Lambda_L} + V(\Lambda_L) F\left(\frac{N_{\Lambda_L}}{V(\Lambda_L)}\right) \tag{34}$$

where F is a twice continuously differentiable, not strictly convex function with $F(0) = 0$ and $\lim_{x\to\infty} F'(x) = +\infty$. Suppose that a phase transition of the first order occurs for the fermion gas, i.e. $\exists \mu_F^1 \in \mathbb{R}$ such that

$$\lim_{\mu\uparrow\mu_F^1} \bar{\rho}_F(\beta, \mu) = \rho_F^1, \qquad \lim_{\mu\downarrow\mu_F^1} \bar{\rho}_F(\beta, \mu) = \rho_F^2 \qquad \text{with } \rho_F^1 < \rho_F^2.$$

Then $\exists \mu_B^1 \in \mathbb{R}$ such that

$$\lim_{\mu\uparrow\mu_B^1} \bar\rho_B(\beta,\mu)=\rho_B^1<\rho_F^1<\rho_F^2<\rho_B^2=\lim_{\mu\downarrow\mu_B^1}\bar\rho_B(\beta,\mu).$$

Proof. Since there is a phase transition of first order at $\mu=\mu_F^1$ for the fermion gas, it follows that

$$\forall x\in\mathbb{R}^+: A_F(x,\mu_F^1)\geqslant A_F(\rho_F^1,\mu_F^1)=A_F(\rho_F^2,\mu_F^1)\equiv B \tag{35}$$

and

$$\frac{\partial}{\partial x}A_F(x,\mu_F^1)\bigg|_{\rho_F^1}=\frac{\partial}{\partial x}A_F(x,\mu_F^1)\bigg|_{\rho_F^2}=0$$

or

$$\mu_F^I(\rho_F^1)+F'(\rho_F^1)-\mu_F^1=\mu_F^I(\rho_F^2)+F'(\rho_F^2)-\mu_F^1=0. \tag{36}$$

We have to prove that for all $\mu\in\mathbb{R}$, $A_B(x,\mu)$ never attains its minimum value in the interval $[\rho_F^1,\rho_F^2]$, which means that for all $\mu\in\mathbb{R}$ and for all $x\in[\rho_F^1,\rho_F^2]$, $\bar\rho_B(\beta,\mu)\neq x$, or, that there exists a μ_B^1 such that

$$\lim_{\mu\uparrow\mu_B^1}\bar\rho_B(\beta,\mu)=\rho_B^1<\rho_F^1 \qquad \text{and} \qquad \lim_{\mu\downarrow\mu_B^1}\bar\rho_B(\beta,\mu)=\rho_B^2>\rho_F^2.$$

Let $\mu_B^2=\mu_F^1-(\mu_F^I(\rho_F^2)-\mu_B^I(\rho_F^2))$ and for later use, $\mu_B^3=\mu_F^1-(\mu_F^I(\rho_F^1)-\mu_B^I(\rho_F^1))$. Then

$$\mu_B^2<\mu_B^3 \qquad \text{and} \qquad \bar\rho_B(\beta,\mu_B^2)<\bar\rho_B(\beta,\mu_B^3).$$

The strategy of the proof consists in first proving that there are two cases to consider: $\rho_F^2<\bar\rho_B(\beta,\mu_B^2)$ or $\bar\rho_B(\beta,\mu_B^2)<\rho_F^1$. Then, for each of these possibilities, we prove that for all $\mu\in\mathbb{R}$ and for all $\rho\in[\rho_F^1,\rho_F^2]$ we have $\rho\neq\bar\rho_B(\beta,\mu)$.

Now, we proceed to the proof of the first statement. We look for the minimum of $A_B(x,\mu_B^2)\equiv B(x)$.

$$\begin{aligned}B'(\rho_F^1)&=\mu_B^I(\rho_F^1)+F'(\rho_F^1)-\mu_F^1+(\mu_F^I(\rho_F^2)-\mu_B^I(\rho_F^2))\\&=(\mu_F^I(\rho_F^2)-\mu_B^I(\rho_F^2))-(\mu_F^I(\rho_F^1)-\mu_B^I(\rho_F^1))>0\end{aligned}$$

where we used (36) to obtain the second equality. Thus B is strictly increasing at ρ_F^1, which implies that there exists an $x_1<\rho_F^1$ with

$$B(x_1)<B(\rho_F^1). \tag{37}$$

Furthermore, define

$$C(x)=A_F(x,\mu_F^1)-B(x).$$

Then

$$C'(x)=(\mu_F^I(x)-\mu_B^I(x))-(\mu_F^I(\rho_F^2)-\mu_B^I(\rho_F^2))$$

and $C'(x)\leqslant 0$ for all $x\leqslant\rho_F^2$. Therefore

$$A_F(\rho_F^1,\mu_F^1)-B(\rho_F^1)\geqslant A_F(\rho,\mu_F^1)-B(\rho) \qquad \text{for all } \rho\in[\rho_F^1,\rho_F^2]$$

or, using (35),

$$B(\rho)\geqslant B(\rho_F^1)+A_F(\rho,\mu_F^1)-A_F(\rho_F^1,\mu_F^1)\geqslant B(\rho_F^1) \qquad \text{for all } \rho\in[\rho_F^1,\rho_F^2]. \tag{38}$$

Combining (37) and (38), we find

$$\exists x_1 < \rho_F^1 \text{ such that for all } \rho \in [\rho_F^1, \rho_F^2]\colon B(x_1) < B(\rho_F^1) \leqslant B(\rho). \tag{39}$$

This means that the function $B(x) = A_B(x, \mu_B^2)$ does not attain its minimum value at any $\rho \in [\rho_F^1, \rho_F^2]$. So, $\bar\rho_B(\beta, \mu_B^2) > \rho_F^2$ or $\bar\rho_B(\beta, \mu_B^2) < \rho_F^1$.

Next, we prove the second statement.

(1) If $\rho_B(\beta, \mu_B^2) > \rho_F^2$, we distinguish between (a) $\mu \geqslant \mu_B^2$, (b) $\mu < \mu_B^2$.

(a) $\mu \geqslant \mu_B^2$: as $\rho_B(\beta, \mu)$ is an increasing function of μ, one has

$$\bar\rho_B(\beta, \mu) \geqslant \bar\rho_B(\beta, \mu_B^2) > \rho_F^2, \qquad \forall \mu > \mu_B^2.$$

(b) $\mu < \mu_B^2$: using (39)

$$\begin{aligned} A_B(x_1, \mu) &= A_B(x_1, \mu_B^2) + (\mu_B^2 - \mu)x_1 = B(x_1) + (\mu_B^2 - \mu)x_1 < B(\rho_F^1) + (\mu_B^2 - \mu)\rho_F^1 \\ &= A_B(\rho_F^1, \mu) \leqslant B(\rho) + (\mu_B^2 - \mu)\rho = A_B(\rho, \mu) \qquad \text{for all } \rho \in [\rho_F^1, \rho_F^2]. \end{aligned} \tag{40}$$

This means that for all $\mu < \mu_B^2 : A_B(x_1, \mu) < A_B(\rho, \mu), \forall \rho \in [\rho_F^1, \rho_F^2]$. Combining the results of (a) and (b), we get that for all $\mu \in \mathbb{R}$ and $\rho \in [\rho_F^1, \rho_F^2]$, $\bar\rho_B(\beta, \mu) \neq \rho$ holds.

(2) If $\rho_B(\beta, \mu_B^2) < \rho_F^1$, we will again distinguish two cases. Therefore, we look for the minimum of $A_B(x, \mu_B^3) \equiv D(x)$. Then

$$D'(\rho_F^2) = (\mu_F^I(\rho_F^1) - \mu_B^I(\rho_F^1)) - (\mu_F^I(\rho_F^2) - \mu_B^I(\rho_F^2)) < 0$$

which implies that

$$\exists y_1 > \rho_F^2 \text{ such that} \qquad D(y_1) < D(\rho_F^2). \tag{41}$$

Also, let $E(x) = A_F(x, \mu_F^1) - D(x)$. Then

$$E'(x) = (\mu_F^I(x) - \mu_B^I(x)) - (\mu_F^I(\rho_F^1) - \mu_B^I(\rho_F^1))$$

and $E'(x) > 0$ for all $x > \rho_F^1$.

Hence, analogously as for $A_B(x, \mu_B^2)$, one finds

$$D(\rho) \geqslant D(\rho_F^2) \qquad \text{for all } \rho \in [\rho_F^1, \rho_F^2]. \tag{42}$$

Combining (41) and (42) yields

$$\exists y_1 > \rho_F^2 \text{ such that for all } \rho \in [\rho_F^1, \rho_F^2]\colon D(\rho) \geqslant D(\rho_F^2) > D(y_1). \tag{43}$$

This means that either $\bar\rho_B(\beta, \mu_B^3) < \rho_F^1$ or $\bar\rho_B(\beta, \mu_B^3) > \rho_F^2$. If $\bar\rho_B(\beta, \mu_B^3) < \rho_F^1$ we again distinguish (a) $\mu \leqslant \mu_B^3$, (b) $\mu > \mu_B^3$.

(a) $\mu \leqslant \mu_B^3 : \rho_B(\beta, \mu)$ is an increasing function of μ; therefore,

$$\bar\rho_B(\beta, \mu) \leqslant \bar\rho_B(\beta, \mu_B^3) < \rho_F^1 \qquad \text{for all } \mu \leqslant \mu_B^3.$$

(b) $\mu > \mu_B^3$: using (43), one finds as in (1b)

$$A_B(\rho, \mu) \geqslant A_B(\rho_F^2, \mu) > A_B(y_1, \mu), \qquad \text{for all } \rho \in [\rho_F^1, \rho_F^2] \text{ and all } \mu > \mu_B^3.$$

(a) and (b) again yield the result.

The last case we have to consider is $\bar\rho_B(\beta, \mu_B^2) < \rho_F^1$ and $\bar\rho_B(\beta, \mu_B^3) > \rho_F^2$. Then

$$\forall \mu < \mu_B^2 : \bar\rho_B(\beta, \mu) < \rho_F^1, \tag{44}$$

$$\forall \mu > \mu_B^3 : \bar\rho_B(\beta, \mu) > \rho_F^2. \tag{45}$$

We still have to prove that for all $\mu \in]\mu_B^2, \mu_B^3[$ and for all $\rho \in [\rho_F^1, \rho_F^2]$, $\bar\rho_B(\beta, \mu) \neq \rho$. Take any $\mu \in]\mu_B^2, \mu_B^3[$ and choose $\rho_0 \in \mathbb{R}$ such that $\mu = \mu_F^1 - (\mu_F^I(\rho_0) - \mu_B^I(\rho_0))$. It can easily be seen that $\rho_F^1 < \rho_0 < \rho_F^2$. Let $G(x) = A_B(x, \mu)$ and $H(x) = A_F(x, \mu_F^1) - G(x)$. Then,

$$G'(\rho_F^1) = (\mu_F^I(\rho_0) - \mu_B^I(\rho_0)) - (\mu_F^I(\rho_F^1) - \mu_B^I(\rho_F^1)) > 0,$$

$$G'(\rho_F^2) = (\mu_F^I(\rho_0) - \mu_B^I(\rho_0)) - (\mu_F^I(\rho_F^2) - \mu_B^I(\rho_F^2)) < 0,$$

implying that $\exists x_2, y_2 \in \mathbb{R}^+$ such that $x_2 < \rho_F^1 < \rho_F^2 < y_2$ and

$$G(x_2) < G(\rho_F^1), \tag{46}$$

$$G(y_2) < G(\rho_F^2). \tag{47}$$

Also $H'(x) = (\mu_F^I(x) - \mu_B^I(x)) - (\mu_F^I(\rho_0) - \mu_B^I(\rho_0))$ and $H'(x) \leqslant 0$, $\forall x \leqslant \rho_0$, $H'(x) \geqslant 0$, $\forall x \geqslant \rho_0$. So, as above, we find

$$\exists x_2 < \rho_F^1 \text{ such that for all } \rho \in [\rho_F^1, \rho_0]: G(x_2) < G(\rho_F^1) \leqslant G(\rho), \tag{48}$$

$$\exists y_2 > \rho_F^2 \text{ such that for all } \rho \in [\rho_0, \rho_F^2]: G(\rho) \geqslant G(\rho_F^2) > G(y_2). \tag{49}$$

Combining (44), (45), (48) and (49), we obtain the result.

As a byproduct, we obtain immediately from theorem 3.6 the following result.

Corollary 3.7. Let $(T_c)_{B(F)}$ be the temperature above which there is no phase transition of the first order. Then $(T_c)_B \geqslant (T_c)_F$.

Proof. For any $T \leqslant (T_c)_F$ such that there is a phase transition for the fermion gas, there is also a phase transition for the boson gas. Therefore $(T_c)_F \leqslant (T_c)_B$.

Proposition 3.8. With the Hamiltonian (34), one has that (i) $p_{B(F)}(\beta, \mu)$ is an increasing and convex function of μ and therefore continuous. (ii) $p_{B(F)}(\beta, \mu)$ is everywhere differentiable with derivative $dp_{B(F)}(\beta, \mu)/d\mu = \bar{\rho}_{B(F)}(\beta, \mu)$ except at the points of discontinuity of the map $\mu \to \bar{\rho}_{B(F)}(\beta, \mu)$.

Proof. (i) follows from the convexity of $p_{B(F)}(\Lambda_n, \beta, \mu)$ and $\lim_{n\to\infty} p_{B(F)}(\Lambda_n, \beta, \mu) = p_{B(F)}(\beta, \mu)$ (lemma 3.5). (ii) follows directly from the lemma of Griffiths (Hepp and Lieb 1973), and the remark that $\lim_{L\to\infty} \rho_{B(F)}(\Lambda_L, \beta, \mu) = \bar{\rho}_{B(F)}(\beta, \mu)$ for all μ, points of continuity.

Before, we always had a one-to-one correspondence between μ and the fermion density. However, when phase transitions of the first order occur, this is not the case anymore. Therefore, it is more appropriate to define $p_{B(F)}$ as a function of β and ρ instead of β and μ. If we want to denote the pressure as a function of μ, we use further on the notation $\bar{p}_{B(F)}(\beta, \mu)$.

Let $I_{B(F)}$ be the countable index set labelling the set of discontinuity points of the map $\mu \to \bar{\rho}_{B(F)}(\beta, \mu)$. We denote with $\mu_{B(F)}^i$, $i \in I_{B(F)}$, the discontinuity points. At these points

$$\lim_{\mu \uparrow \mu_{B(F)}} \bar{\rho}_{B(F)}(\beta, \mu) = \rho_{B(F)}^{1i}, \qquad \lim_{\mu \downarrow \mu_{B(F)}} \bar{\rho}_{B(F)}(\beta, \mu) = \rho_{B(F)}^{2i}.$$

As already mentioned, if $\rho \in [\rho_{B(F)}^{1i}, \rho_{B(F)}^{2i}]$ and $\mu_{B(F)}^{\Lambda_n}$ is defined by $\rho_{B(F)}(\Lambda_n, \beta, \mu_{B(F)}^{\Lambda_n}) = \rho$, then

$$\lim_{n\to\infty} \mu^{\Lambda_n}_{B(F)} = \mu^i_{B(F)}.$$

Therefore, from proposition 3.8, it follows that for all $i \in I_{B(F)}$ and all

$$\rho \in [\rho^{1i}_{B(F)}, \rho^{2i}_{B(F)}] : p_{B(F)}(\beta, \rho) = \tilde{p}_{B(F)}(\beta, \mu^i_{B(F)}).$$

Finally, remark that if $\bar{\rho}_{B(F)}(\beta, \mu)$ is differentiable with respect to μ at a point μ_0 and $\rho = \bar{\rho}_{B(F)}(\beta, \mu_0)$, then

$$\left.\frac{dp_F(\beta, \rho)}{d\rho}\right|_\rho = \left.\frac{\rho}{d\bar{\rho}_{B(F)}(\beta, \mu)/d\mu}\right|_{\mu_0}. \tag{50}$$

Theorem 3.9. With the Hamiltonian (34) and with the conventions for the pressure one has

$$p_B(\beta, \rho) < p_F(\beta, \rho), \qquad \forall \beta \in \mathbb{R}^+_0, \forall \rho > 0.$$

Proof. (i) Suppose there are no phase transitions for the boson gas (or $T > (T_c)_B$ where $(T_c)_B$ is defined as in corollary 3.7).

Then the proof goes along the same lines as in theorem 3.4. (ii) $T \leqslant (T_c)_B$: suppose there are phase transitions for $\mu = \mu^i_B$ ($i \in I_B$), where $\bar{\rho}_B(\beta, \mu)$ jumps from ρ^{1i}_B to ρ^{2i}_B.

If $\rho^{1i}_B \leqslant \rho \leqslant \rho^{2i}_B$, we have:

(a) if there is a phase transition for the fermion gas, i.e. $\bar{\rho}_F(\beta, \mu)$ jumps from ρ^1_F to ρ^2_F at μ^1_F, then $\exists j \in I_B$ such that $\rho^{1j}_B < \rho^1_F < \rho^2_F < \rho^{2j}_B$ (theorem 3.6);

(b) $p_{B(F)}(\beta, \rho)$ is constant for all $\rho \in [\rho^{1i}_{B(F)}, \rho^{2i}_{B(F)}]$. Therefore, $p_B(\beta, \rho) < p_F(\beta, \rho)$ for all $\rho \in [\rho^{1i}_B, \rho^{2i}_B]$ if $p_B(\beta, \rho^{1i}_B) < p_F(\beta, \rho^{1i}_B)$.

If $\rho \notin [\rho^{1i}_B, \rho^{2i}_B]$ for all $i \in I_B$ and $\rho \neq \rho_c$ where ρ_c is the critical density for boson condensation, it can be proved as in theorem 3.4 that

$$\left.\frac{d\bar{\rho}_B(\beta, \mu)}{d\mu}\right|_{f(\mu_F)} > \left.\frac{d\bar{\rho}_F(\beta, \mu)}{d\mu}\right|_{\mu_F}$$

where the function f is as above.

Again, for $\rho = \rho_c$ special care should be taken and we have to use an argument similar to the one in theorem 2.2. Using this, we can again apply our argument as in theorem 2.1 to obtain the result. The picture of $p_{B(F)}$ as a function of the density ρ in the case of phase transitions looks as in figure 1. If $F(x) = -ax^2 + bx^4$ it can be seen that there is maximally one platform where $p_{B(F)}$ is constant as a function of ρ. In this case the phase diagram resembles the phase diagram of water (if we don't take into account the solid phase).

We treated here imperfect models with a Hamiltonian as in (34), where F is twice continuously differentiable everywhere. The results do not change if F is twice continuously differentiable almost everywhere with respect to the Lebesgue measure. In this case, phase transitions of the second order may also occur. The proof of theorem 3.9 goes along the same lines, only some care should be taken for those points ρ where a phase transition of second order occurs.

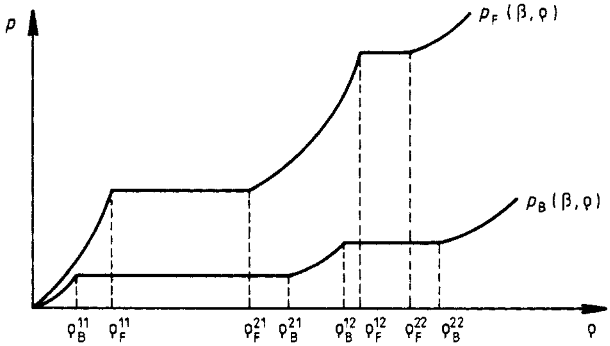

Figure 1. The pressure as a function of the density for general mean field interactions.

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